Eisenstein series - take II
higher rank

http://www.math.huji.ac.il/~erezla

June 19, 2006

Let G be a split reductive group over $\mathbb{Q}$. $G(\mathbb{A})^1 = \cap_{\chi \in X^*(G)} \ker |\chi|$ where $X^*(G)$ is the lattice of characters of G. For example, $G = GL_n$, $G(\mathbb{A})^1 = \{g \in GL_n(\mathbb{A}) : |\det g| = 1\}$. We always have $\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1) < \infty$ and $G(\mathbb{A})^1 \backslash G(\mathbb{A}) \simeq \mathbb{R}^k_{>0}$ for some k. Given f on $G(F)\backslash G(\mathbb{A})$ and a parabolic $P = MU$ the constant term along P is defined by

$$f_P(g) = \int_{U(F)\backslash U(\mathbb{A})} f(ug)\ du.$$

It is a function on $M(F)U(\mathbb{A})\backslash G(\mathbb{A})$. Define

$$L^2_{cusp}(G(F)\backslash G(\mathbb{A})^1) = \{\varphi \in L^2(G(F)\backslash G(\mathbb{A})^1) : f_P \equiv 0 \text{ for all proper parabolic } P\}$$

FACT: $L^2_{cusp}(G(F)\backslash G(\mathbb{A})^1)$ decomposes discretely. The theory of Eisenstein series reduces, in some sense, the study of $L^2(G(F)\backslash G(\mathbb{A}))$ to the study of $L^2_{cusp}(M(F)\backslash M(\mathbb{A}))$ for all Levi subgroup M of G.

Let $\mathcal{A}^0_G = \mathcal{A}^0(G(F)\backslash G(\mathbb{A})^1)$ denote the "algebraic" part of $L^2_{cusp}(G(F)\backslash G(\mathbb{A})^1)$: functions which span a finite length subrepresentation ($\mathfrak{z}$-finite) and which are K-finite. Thus, $\mathcal{A}_G^0 = \oplus_\pi \mathcal{A}_{G,\pi}^0$, where π range over cuspidal representations of $G(\mathbb{A})$.

cuspidal Eisenstein series - maximal parabolic case

Let $P = MU$ maximal parabolic. Also fix a "good" maximal compact $K = \prod_v K_v$ of $G(\mathbb{A})$. Let T_M be the center of M in the derived group of G. It is isomorphic to the multiplicative group $\mathbb{G}_m$. Let A_M be $\mathbb{R}_{>0}$ imbedded in $T_M(\mathbb{A})$. Let ϖ be the fundamental weight corresponding to P. It can be viewed as a rational multiple of a character of M. We get a quasi-character $|\varpi| : M(F)\backslash M(\mathbb{A}) \to \mathbb{R}_{>0}$ For example, if $G = GL_n$, $M = GL_{n_1} \times GL_{n_2}$, $n_1 + n_2 = n$, $K = O(n, \mathbb{R}) \prod_{p<\infty} GL_n(\mathbb{Z}_p)$ and $|\varpi|(g_1, g_2) = \left(\frac{|\det g_1|^{1/n_1}}{|\det g_2|^{1/n_2}}\right)^{\frac{1}{2}}$.

We denote by $\mathcal{A}^0_P = \mathcal{A}^0(M(F)U(\mathbb{A})\backslash G(\mathbb{A}))$ the space of functions $\varphi : G(\mathbb{A}) \to \mathbb{C}$ such that $\varphi(amug) = \delta_P(a)^{\frac{1}{2}}\varphi(g)$ for $m \in M(F)$, $u \in U(\mathbb{A})$, $a \in A_M$ and for all $k \in K$ the function $m \mapsto \varphi(mk) \in \mathcal{A}^0_M$. We have $\mathcal{A}^0_P = \oplus_\pi \mathcal{A}^0_{P,\pi}$ (sum over cuspidal π of M).

We can identify $\mathcal{A}^0_P = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{A}^0_M$ by

$$\varphi \mapsto f(g) = \delta_P^{-\frac{1}{2}}\varphi(\cdot g)$$
$$f \mapsto \varphi(g) = [f(g)](m)$$

Set

$$\varphi_s(g) = |\varpi|^s\,(m)\varphi(mk)$$

for $g = umk$, $u \in U(\mathbb{A})$, $m \in M(\mathbb{A})$, $k \in K$ and $s \in \mathbb{C}$. With the action $I(g,s)\varphi = (\varphi_s(\cdot g))_{-s}$ $\mathcal{A}^0_P$ becomes $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{A}^0_M \otimes |\varpi|^s$. We write $\mathcal{A}^0_P(s)$ the space $\mathcal{A}^0_P(s)$ with the action $I(g,s)$. The isomorphism $\mathcal{A}^0_{M,\pi} \simeq \mathrm{Hom}_{M(\mathbb{A})}(\pi, \mathcal{A}^0_M) \otimes \pi$ gives rise to $\mathcal{A}^0_{P,\pi} \simeq \mathrm{Hom}_{M(\mathbb{A})}(\pi, \mathcal{A}^0_M) \otimes I_P(\pi, s)$.

Define

$$E(g, \varphi, s) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi_s(\gamma g).$$

Properties:

series is absolutely convergent for $\mathrm{Re}(s) \gg 0$.

admits meromorphic continuation to the whole plane and a functional equation.

only finitely many poles for $\mathrm{Re}(s) > 0$, all simple and they lie on the real line. The residues are in $L^2(G(F)\backslash G(\mathbb{A})^1)$.

holomorphic for $\mathrm{Re}(s) = 0$.

Whenever regular, $E(\cdot, \varphi, s)$ defines an intertwining map $\mathcal{A}_P^0(s) \to \mathcal{A}_G$ (automorphic forms on G).

Let $\bar{P} = M\bar{U}$ be the opposite parabolic. Then $E_{\bar{P}} = M(s)\varphi$ ($+\varphi_s$, if $\bar{P}$ is conjugate to P) where

$$M(s)\varphi = \int_{\bar{U}(\mathbb{A})} \varphi_s(ug)\ du.$$

The latter is an intertwining operator $\mathcal{A}^0_P(s) \to \mathcal{A}^0_{\bar{P}}(-s)$. (Note that $\varpi_{\bar{P}} = \varpi_P^{-1}$.) $M(s)$ admits similar analytic properties.

The functional equations are

$$E(g, M(s)\varphi, -s) = E(g, \varphi, s)$$
$$M(\bar{P}, -s) \circ M(P, s) = \mathrm{Id}$$

Remarks about proofs:

Key Fact: for $\mathrm{Re}(s) \gg 0$, $E(\cdot, \varphi, s)$ is the unique form satisfying

$$F_P = \varphi_s$$
$$F_{\bar{P}} \text{ has exponent } -s$$
$$F_Q \equiv 0 \text{ for } Q \neq P, \bar{P}$$

One direction is a straightforward computation. The other uses L^2-argument.

The Key Fact implies the meromorphic continuation and the functional equation. It also implies that the residues are in L^2.

$M(P,s)^* = M(\bar{P},\bar{s})$, and therefore the functional equation implies that $M(s)$ is unitary for $\operatorname{Re} s = 0$, therefore holomorphic.

Maass-Selberg relations: for $s = \sigma + \mathrm{it}$

$$\|\Lambda^T E(\cdot,\varphi,s)\|_2^2 = \frac{e^{\sigma T}(\varphi,\varphi) - e^{-\sigma T}(M(s)\varphi, M(s)\varphi)}{\sigma} + \frac{\operatorname{Im} e^{\mathrm{itT}}(M(s)\varphi,\varphi)}{t}$$

where

$$(\varphi_1,\varphi_2) = \int_{A_M M(F)U(\mathbb{A})\backslash G(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\, dg$$

Consequences:
$E(\varphi,s)$ is holomorphic whenever $M(s)$ is.

Also, for $\sigma > 0$, $t \neq 0$ $M(s)$ has no pole, otherwise $-e^{-\sigma T}(M(s)\varphi, M(s)\varphi)/\sigma$ will dominate the RHS, in contradiction to the fact that LHS is non-negative. Similarly, for $t = 0$ a pole must be simple for the same reason.

The meromorphic continuation works for smooth functions (not necessarily K-finite). However it does not hold for non-$\mathfrak{z}$-finite functions!

Langlands' insight:

On $\mathcal{A}^0_{P,\pi} = \mathrm{Hom}(\pi, \mathcal{A}^0_{M,\pi}) \otimes I_P(\pi, s)$ $M(s) = \mathrm{Id} \otimes M(\pi, s)$ where $M(\pi, s) : I_P(\pi, s) \to I_{\bar{P}}(\pi, s)$. If $\pi = \otimes \pi_p$, $M(\pi, s) = \otimes M_p(\pi_p, s)$.

Suppose that π_p is unramified, i.e. it has a vector fixed under $G(\mathbb{Z}_p)$. This happens for almost all p and such a vector is unique up to a scalar. Moreover, π_p is a subquotient of $\mathrm{Ind}_{B_M(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)} \chi_p$ for some unramified character χ_p of $T_0(\mathbb{Q}_p)$.

(Fix a maximal torus T_0 of G and a Borel subgroup B containing T_0; $B_M = B \cap M$.) The orbit of χ_p under the Weyl group of M is determined by π_p. On the normalized unramified vector $M(\pi_p, s)$ is given by the scalar

$$\prod_\alpha \frac{1 - p^{-1}(\chi_p\,|\varpi|^s)(\alpha^\vee(p))}{1 - (\chi_p\,|\varpi|^s)(\alpha^\vee(p))}$$

where the product is over all roots α of T_M in U. (Gindikin-Karpelevic formula - analogue from the real case.) We can further decompose this product as

$$\prod_j \prod_{\alpha:\langle\varpi,\alpha^\vee\rangle=j} \frac{1 - \chi_p(\alpha^\vee(p))p^{-js-1}}{1 - \chi_p(\alpha^\vee(p))p^{-js}} \qquad (1)$$

where α ranges over the roots such that in the decomposition of $\alpha^\vee$ into a linear combination of simple co-roots, the co-root corresponding to P appears with coefficient j.

Recall that unramified representations of $G(\mathbb{Q}_p)$ are parameterized by Weyl orbits of unramified characters χ_p of $T_0(\mathbb{Q}_p)$. The latter are determined by their values at $\psi_j(p)$ where $\psi_1, \ldots, \psi_r$ is a basis for the lattice of co-characters $X_*(T_0)$ of rational characters of T_0, which is the dual lattice to $X^*(T_0)$. Thus, χ_p can be viewed as a element of $\text{Hom}(X_*(T_0), \mathbb{C}^*) = X^*(T_0) \otimes \mathbb{C}^*$. Now, we can define a complex group ${}^LG$ ("dual" to G) with torus ${}^LT_0$ such that $X_*({}^LT_0) = X^*(T_0)$ and the roots of T_0 are the co-roots of ${}^LT_0$. We can identify ${}^LT_0(\mathbb{C})$ with $X_*({}^LT_0) \otimes \mathbb{C}^* = X^*(T_0) \otimes \mathbb{C}^*$. The semi-simple classes of ${}^LG(\mathbb{C})$ are identified with ${}^LT_0(\mathbb{C})/W = X^*(T_0) \otimes \mathbb{C}^*/W$. Thus, to an unramified representation τ_p of $G(\mathbb{Q}_p)$ corresponds a semi-simple conjugacy class $\lambda(\tau_p)$ of ${}^LG(\mathbb{C})$ which is called the Frobenius-Hecke parameter of τ_p.

In ${}^LG$ there is a corresponding parabolic ${}^LP$ with Levi ${}^LM$ and unipotent ${}^LU$. If λ_p is the parameter of π_p (a semisimple class in ${}^LM$) then the j-th factor in (1) is

$$\frac{\det(1 - p^{-js-1}\tilde{r}_j(\lambda_p))}{\det(1 - p^{-js}\tilde{r}_j(\lambda_p))}$$

where r_j is the adjoint representation of ${}^LM$ on the part of $\mathcal{L}({}^LU)$ on which ϖ acts by j. Thus $\oplus_{j=1}^k r_j$ is the decomposition into irreducibles of Ad of ${}^LM$ on $\mathcal{L}({}^LU)$; r_j corresponds to the j-th constituent in the lower central series for ${}^LU$.

Thus, on $\mathcal{A}^0_{\pi,s}$ $M(s)$ is given roughly as

$$\prod_{j=1}^{k} \frac{L(js, \pi, \tilde{r}_j)}{L(js+1, \pi, \tilde{r}_j)}$$

where for a representation $r : {}^LG \to GL_n(\mathbb{C})$

$$L(s, \pi, r) = \prod_p \det(1 - p^{-s} r(\lambda(\pi_p)))^{-1} \quad \mathrm{Re}(s) \gg 0.$$

This implies the meromorphic continuation (strip by strip) of the $L(s, \pi, r_j)$'s.

To summarize:

- From the structure of the Gindikin-Karpelevic formula come up with the L-group

- General notion of L-functions
- Conjecture general analytic properties of L-functions
- functoriality conjecture: any L-function (i.e. an Euler product with expected nice analytic properties) is an automorphic L-function of GL_n.

These revolutionary ideas came (in a domino effect) from the computation of the constant term of cuspidal Eisenstein series, carried out by Langlands in the 60's (Euler products manuscript, Yale University). It is therefore a turning point in the theory of automorphic forms.

Examples: $G = GL_n$, $M = GL_{n_1} \times GL_{n_2}$, $\pi = \pi_1 \otimes \pi_2$, π_i a cuspidal representation of $GL_{n_i}(\mathbb{A})$.

In this case $k = 1$ and $r_1 : GL(\mathbb{C}^{n_1}) \times GL(\mathbb{C}^{n_2}) \to GL(\mathbb{C}^{n_1} \otimes (\mathbb{C}^{n_2})^*)$ is the tensor product representation. It gives rise to Rankin-Selberg L-functions.

$G = Sp_{2(n+m)}$, $M = GL_n \times Sp_{2m}$, $\pi = \tau \otimes \sigma$, ${}^LM = GL_n(\mathbb{C}) \times SO_{2m+1}(\mathbb{C})$. In this case $k = 2$. $r_1 : {}^LM \to GL_{n(2m+1)}(\mathbb{C})$ is the tensor product representation of $GL_n(\mathbb{C}) \times GL_{2m+1}(\mathbb{C})$ restricted to ${}^LM$ while $r_2 : {}^LM \to GL_{\binom{n}{2}}(\mathbb{C})$ is the exterior square representation $\wedge^2$ of $GL_n(\mathbb{C})$. (In the case $m = 0$ only the latter appears.) Similarly for other classical groups.

Exceptional cases $G = G_2$; one of the maximal parabolic subgroups gives rise to the 4-dimension symmetric cube representation of $GL_2(\mathbb{C})$.

$G = E_8$, $M = GL_2 \times GL_3 \times GL_5$ (up to isogeny). Here $k = 6$ and

$$r_1 : GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \times GL_5(\mathbb{C}) \to GL_{30}(\mathbb{C})$$

in the (triple) tensor product representation.

$G = E_8$, $M = GL_5 \times GL_4$ (up to isogeny). Here $k = 5$ and $r_1 : GL_5(\mathbb{C}) \times GL_4(\mathbb{C}) \to GL_{40}(\mathbb{C})$ is given by $r_1(g_1, g_2) = \wedge^2(g_1) \otimes g_2$.

$G = E_8$, $M = GL_8$; $k = 3$ and r_1 is the exterior cube (56-dimensional) representation of $GL_8(\mathbb{C})$.

$G = E_8$, $M = Spin(14)$; $k = 2$ and r_1 is the (64-dimensional) spin representation of $Spin(14)$.

General case $P = MU$ any parabolic, T_M the center of M in the derived group of G (a torus of dimension r – the co-rank of M), $A_M \simeq \mathbb{R}^r_{>0}$ the positive real points in $T_M(\mathbb{A})$, $X^*(M)$ - the lattice (of rank r) of characters of M trivial on the center of G. We form the vector space $\mathfrak{a}^*_M = X^*(M) \otimes \mathbb{R} = X^*(T_M) \otimes \mathbb{R}$; $\mathfrak{a}^*_{M,\mathbb{C}} = X^*(M) \otimes \mathbb{C}$. This space gives rise to quasi-character $m \mapsto |m|^{\lambda} : M(F)\backslash M(\mathbb{A}) \to \mathbb{C}^*$ defined by

$$|m|^{\chi \otimes s} = |\chi(m)|^s = \prod_{p \leq \infty} |\chi(m_p)|_p^s$$

Consider the space

$$\mathcal{A}_P^2 = \{\varphi : M(F)U(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}| \\ \delta_P(m)^{-\frac{1}{2}}\varphi(mk) \in \mathcal{A}_M^2 \text{ for all } k \in K$$

where $\mathcal{A}_M^2$ is the space of K-finite functions on $\mathbb{A}_M M(F)\backslash M(\mathbb{A})$, which span a finite length representation in $L^2_{disc}(M(F)\backslash M(\mathbb{A})^1)$. Again $\mathcal{A}_P^2 = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{A}_M^2$ and by twisting

$$\varphi_\lambda(umk) = |m|^{\lambda} \varphi(mk)$$

we have $\mathcal{A}_P^2(\lambda) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{A}_M^2 \otimes |\cdot|^{\lambda}$ for any $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. As before, the Eisenstein series are defined by

$$E(g, \varphi, \lambda) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi_\lambda(\gamma g)$$

Whenever regular, they define intertwining maps $\mathcal{A}_P^2(\lambda) \to \mathcal{A}^G$. We also have the intertwining operators

$$M_{P'|P}(\lambda) : \mathcal{A}_P^2(\lambda) \to \mathcal{A}_{P'}^2(\lambda)$$

defined for any P' with Levi M by

$$M_{P'|P}(\lambda)\varphi(g) = \int_{(U_P(\mathbb{A})\cap U_{P'}(\mathbb{A}))\backslash U_{P'}(\mathbb{A})} \varphi_\lambda(ug)\, du$$

Properties

- $E(\varphi,\lambda)$ and $M(\lambda)$ converge for $\mathrm{Re}(\langle \lambda, \alpha^\vee \rangle) \gg 0$ for all $\alpha \in \Delta_P$.

- admit meromorphic continuation to $\mathfrak{a}^*_{M,\mathbb{C}}$ and functional equations

$$E_{P'}(M_{P'|P}(\lambda)\varphi,\lambda) = E_P(\varphi,\lambda)$$
$$M_{P''|P'}(\lambda)M_{P'|P}(\lambda) = M_{P''|P}(\lambda)$$

- The singularities lie on hyperplane $\langle \lambda, \alpha^\vee \rangle = c$, $\alpha \in \Sigma_P$.

- holomorphic for Re$(\lambda) = 0$. $M_{P'|P}(\lambda)$ is unitary there.

The Eisenstein series are the building blocks for the decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$. This is Langlands' theory which consists of two parts:

1. L^2-decomposition using square-integrable Eisenstein series.

2. constructing the square-integrable automorphic forms as residues of Eisenstein series.

The first part is more "rigid" in a sense, but originally it relied on the second part (which was completed first). In fact, even the analytic properties of square-integrable Eisenstein series were not known before the second part. Nowadays, there is an independent proof of this (due to Bernstein), and in principle the first part can be done independently of the second part. (For the local analogue, the Plancherel formula - cf. Waldspurger, following Harish-Chandra.)

More precise statement:
$L^2(G(F)\backslash G(\mathbb{A})) = \oplus_{[M]} L^2_{[M]}(G(F)\backslash G(\mathbb{A}))$ where $[M]$ ranges over classes of Levi subgroups up to association.

Let $\mathcal{P}(M)$ be the class of parabolic subgroups with Levi M (up to conjugation). $L^2_{[M]}(G(F)\backslash G(\mathbb{A}))$ is isometric to the space

$$F = (F_P)_{P \in \mathcal{P}(M)} : \mathrm{i}\mathfrak{a}^*_{\mathrm{M}} \to \oplus_{\mathrm{P} \in \mathcal{P}(\mathrm{M})} \overline{\mathcal{A}}^2_{\mathrm{P}}$$

satisfying $F_{P'}(\lambda) = M_{P'|P}(\lambda)F_P(\lambda)$ for all $P, P' \in \mathcal{P}_M$, $\lambda \in \mathrm{i}\mathfrak{a}_{\mathrm{M}}^*$ and such that

$$\|F\|^2 = \int_{\mathrm{i}\mathfrak{a}_{\mathrm{M}}^*} \|F_P(\lambda)\|^2 \, d\lambda < \infty$$

The isometry is given by $F \mapsto \int_{i\mathfrak{a}_M^*} E_P(\cdot, F(\lambda), \lambda) \, d\lambda$ (independent of $P \in \mathcal{P}(M)$). In the other direction

$$f \mapsto F_P(\lambda) = \sum_{\{\varphi\}} (f, E_P(\cdot, \varphi, \lambda))$$

where φ ranges over an orthonormal basis of $\mathcal{A}_P^2$.

Residual spectrum: Given M and a cuspidal representation π of $M(\mathbb{A})$. Let $\phi : M(F)U(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$ be such that $\delta_P(m)^{-\frac{1}{2}}\phi(mk) \in \mathcal{A}_{M,\pi}^0$ but for which $\phi(amk)$ is compactly supported in A_M for $m \in M(\mathbb{A})^1$. we form

$$\theta_\phi(g) = \sum_{\gamma \in P(F)\backslash G(F)} \phi(\gamma g)$$

It is absolutely convergent and rapidly decreasing. Let

$$\widehat{\phi}(\lambda)(\cdot) = (\int_{A_M} a^{-\lambda}\delta(a)^{-\frac{1}{2}}\phi(a\cdot)\ da)_{-\lambda} \in \mathcal{A}_P^0$$

The functions $\widehat{\phi}$ are Paley-Wiener functions on $\mathfrak{a}^*_{M,\mathbb{C}}$ with values in a finite-dimensional subspace of $\mathcal{A}_P^0$. By Mellin inversion,

$$\theta_\phi(g) = \sum_{\gamma \in P(F)\backslash G(F)} \int_{\mathrm{Re}\,\lambda=\lambda_0} \widehat{\phi}(\lambda)(\gamma g) =$$

$$\int_{\mathrm{Re}\,\lambda=\lambda_0} E(\lambda, \widehat{\phi}(\lambda), \lambda)\ d\lambda.$$

for $\mathrm{Re}\,\lambda \gg 0$.
Let $L^2_{(M,\pi)}$ be the space spanned by θ_ϕ. We write $(M,\pi) \equiv (M',\pi')$ if $M' = gMg^{-1}$ and $\pi' = \pi^g$ for some $g \in G(F)$. Let $[(M,\pi)]$ denote the class of (M,π)

The starting point is:

$$L^2(G(F)\backslash G(\mathbb{A})) = \oplus_{[M,\pi]} L^2_{(M,\pi)}(G(F)\backslash G(\mathbb{A}))$$

To get a finer decomposition, we start with a formula for the inner product of two pseudo Eisenstein series, and shift the contour to $\operatorname{Re}\lambda = 0$. We get a rediual term from each singular hyperplane crossed, given by an integral over one dimension less. We shift the contour to the closest point to the origin. Continuing this procedure we get iterated residues. More formally:

residue operators: Consider the space of meromorphic functions $\mathcal{M}(\mathfrak{a}^*_{M,\mathbb{C}})$ on $\mathfrak{a}^*_{M,\mathbb{C}}$ whose only singularities are along hyperplanes of the form

$$L_{\alpha,c} = \{\lambda \in \mathfrak{a}^*_{M,\mathbb{C}} : \left\langle \lambda, \alpha^\vee \right\rangle = c\} \quad \alpha \in \Delta_P,\ c \in \mathbb{C}$$

Let V be an affine subspace of $\mathfrak{a}^*_{M,\mathbb{C}}$ which is given by intersections of hyperplanes $L_{\alpha,c}$. Let V^0 denote the vector part of V which a subspace of $\mathfrak{a}^*_M$. A residue operator along V is a linear map $\mathrm{Res}_V : \mathcal{M}(\mathfrak{a}^*_{M,\mathbb{C}}) \to \mathcal{M}(V)$ which is given by a linear combination of the following operators. Given a chain $V = V_k \subsetneq V_{k-1} \subsetneq \dots V_0 = \mathfrak{a}^*_{M,\mathbb{C}}$ with $V_{i+1} = V_i \cap L_{\alpha_i,c_i}$ and points $z_i \in V^0_{i-1} \setminus V^0_i$, $i = 1,\dots,k$ we look at the composition of $r_i : \mathcal{M}(V_{i-1}) \to \mathcal{M}(V_i)$ where $r_i f(v) = \mathrm{Res}_{s=0} f(v + sz_i)$. Of course Res_V depends on some choices.

Simplest case:

$$\mathrm{Res}_V f = \prod_{i=1}^{k} (\left\langle \lambda, \alpha_i^\vee \right\rangle - c_i) f(\lambda)|_V$$

for functions such that the right-hand side is holomorphic.

notation: if V is defined over $\mathbb{R}$ (that is $V \cap \mathfrak{a}^*_M \neq 0$) we let $o(V) \in \mathfrak{a}^*_M \cap V$ be the shortest point in V (the "origin" of V).

We also speak about equivalences of subspaces V under the Weyl group, and denote an equivalence class by $[V]$.

General statement: For any (M, π) there exist a finite collection of singular subspaces V of $\mathfrak{a}^*_{M,\mathbb{C}}$ and canonical residue operators Res_V such that V is defined over $\mathbb{R}$ if V is singular;

$$\mathcal{E} = \sum_w \mathrm{Res}_{wV} E(\phi(\lambda), w\lambda) \quad (\in \mathcal{M}(V))$$

is holomorphic for $\mathrm{Re}\,\lambda = o(V)$ and

$$\theta_\phi = \sum_V \int_{\mathrm{Re}\,\lambda = o(V)} \mathrm{Res}_{wV} \sum_w E(\phi(\lambda), w\lambda)$$

The map

$$p_{[V]}\theta\phi = \sum_{V \in [V]} \int_{\mathrm{Re}\,\lambda = o(V)} \mathrm{Res}_{wV} \sum_w E(\phi(\lambda), w\lambda).$$

is a spectral projection (commutes with $G(\mathbb{A})$), and we can identify the image with L^2-sections in a Hilbertian stack over $o(V) + \mathrm{Im}\,V$. In particular, the discrete spectrum corresponds to $V = pt$.

GL_n (Jacquet, Mœglin-Waldspurger) Discrete spectrum appears for $M = GL_m \times \cdots \times GL_m$

(k times) and $\pi \otimes \ldots \pi$. The singular point is $\lambda_0 = (\frac{k-1}{2}, \ldots, -\frac{k-1}{2})$. The residue is given by

$$\lim_{\lambda=(s_1,\ldots,s_k)\to\lambda_0} \prod_{i=1}^{k-1} (s_i - s_{i+1} - 1) E(\varphi(\lambda), \lambda)$$

classical groups:

G_2 example: Recall roots: α (long), β (short), $\alpha+\beta$, $\alpha+2\beta$ $\alpha+3\beta$, $2\alpha+3\beta$. $M = T_0$. In addition to $\rho = 3\alpha + 5\beta$, the two short roots $\alpha+\beta$, $\alpha+2\beta$ are also singular. There is an intertwining operator (not surjective). $N = \otimes N_p : I(\alpha+2\beta) \to I(\alpha+\beta)$. Let $J = \otimes J_p$ be the image.

$$(\text{Res}\, E(\phi_1), \text{Res}\, E(\phi_2)) =$$

$$[(\phi_1(\alpha+\beta), Np\phi_1(\alpha+2\beta)), (\phi_2(\alpha+\beta), Np\phi_2(\alpha+2\beta))]$$

$p\phi$ depends on the germ of ϕ at the point. where $[\cdot,\cdot]$ is the inner product on $I(\alpha+\beta) \oplus J$ given by

$$((f,g),(f',g')) = ((1+E/2)N_2(f+g), f+g)$$

The image of $N_2 = \otimes (N_2)_p$ is $\otimes(\pi_p^+ \oplus \pi_p^-)$, π_p^+ unramified, $E = \otimes E_p$ acts as 1 on π_p^+ and as -2 on π_p^-.

Thus, we get $\oplus_{\varepsilon=(\varepsilon_p)} \otimes \pi_p^{\varepsilon_p}$ the sum is over all ε such that $1 \neq |\{p : \varepsilon_p = -1\}| < \infty$.

Fourier coefficients of Eisenstein series

Producing cuspidal representations from Eisenstein series (Ginzburg-Rallis-Soudry)

Example: Let π be a cuspidal representation of $GL_{2n}(\mathbb{A})$ such that $L(1, \pi, \wedge^2) = \infty$. By the general philosophy this "means" that π is a functorial lift from SO_{2n+1}. (Reason: in the L-group side π is a lift from a subgroup of $GL_{2n}(\mathbb{C})$ on which $\wedge^2$ has a fixed vector, i.e. from $Sp_{2n}(\mathbb{C})$. Note that ${}^L SO_{2n+1} = Sp_{2n}(\mathbb{C})$).

Consider the Eisenstein series on SO_{4n} induced from $\pi\,|\mathrm{det}|^s$. It has a pole at $s = \frac{1}{2}$. Let Ψ be the residue. The Fourier coefficient with respect to the character $\psi_V = \psi(x_{1,2} + \cdots + x_{n-2,n-1} + \frac{1}{2}(x_{n-1,2n} + x_{n-1,2n+1}))$ on the unipotent radical of the parabolic LV with Levi part $L = GL_1^{n-1} \times SO_{2n+2}$ defines an automoprhic form Φ on SO_{2n+1}, the stabilizer of ψ_V in L.

Example: $n = 2$

$$\Phi(h) = \int_{(F\backslash\mathbb{A})^6} \Psi(\begin{pmatrix} 1 & v & * \\ & h & v^* \\ & & 1 \end{pmatrix})\psi(v)\,dv \quad h \in SO_5(\mathbb{A})$$

for an appropriate character $\psi : (F\backslash\mathbb{A})^6 \to \mathbb{C}$.

It turns out that the Φ's obtained this way form an irreducible cuspidal representation τ of $SO_{2n+1}(\mathbb{A})$ which is generic (has a non-zero non-degenerate Fourier coefficient) and whose functorial transfer to GL_{2n} is π. That is, if $\tau = \otimes_v \tau_v$ and $\tau_v = \mathrm{Ind}(|\cdot|^{s_1}, \ldots, |\cdot|^{s_n})$ at a place where τ_v is unramified then
$\pi_v = \mathrm{Ind}(|\cdot|^{s_1}, \ldots, |\cdot|^{s_n}, |\cdot|^{-s_n}, \ldots, |\cdot|^{-s_1})$.